Linear equation in cause and effect the influence function

Defn:

In the effect at x due to a unit cause concentrat at ξ is denoted by the function $G(x,\xi)$, then the diff effect at x due to uniform distribution of cause of intensity $c(\xi)$ over an elementary region $(\xi + \xi + d\xi)$ is given by $c(\xi)\, G(x,\xi)\, d\xi$

Defn:

The effect $e(x)$ at x due to a distribution cau $c(\xi)$ over the entire region R is given by the integra

$$e(x) = \int_R G(x,\xi)\, c(\xi)\, d\xi \quad \text{--- (1)}$$

Defn:

The function $G(x,\xi)$ which represent the effect at x due to a concentrated cause at ξ is known a the influence function of the problem

Note: 1

1) If the distribution of cause is prescribed if t influence function is known, then

$$e(x) = \int_R G(x,\xi)\, c(\xi)\, d\xi$$ permits the determination

of the effect by direct integration

2) consider a linear relation of the form $c(x) = \phi + \lambda e(x)$ where ϕ is the given function or zero and λ is a constant using $e(x) = \int_R G(x,\xi)\, c(\xi)\, d\xi$ --- (1)

$$c(x) = \phi(x) + \lambda e(x) \quad \text{--- (2)}$$

we get $$c(x) = \phi(x) + \lambda \int G(x,\xi)\, c(\xi)\, d\xi \quad \text{--- (3)}$$

Thus relation is a Fredholm integral eqn of the second kind

using (2) in (1) we get, $e(x) = \int_R G(x,\xi)\,[\phi(\xi) + \lambda e(\xi)]$

$$E(x) = \int_R G(x,\xi)\,\phi(\xi)\,d\xi + \lambda \int_R G(x,\xi)\,e(\xi)\,d\xi \quad \text{--- (4)}$$

Both cause and effect are determined either (3) (4) using (2)

Problem: 1

consider a study of small deflections of a string fixed at a point $x=0$ and $x=l$. under loading distribution of intensity $p(x)$. Find the influence function and deflection

Soln

Suppose that the string is initially so tightly stretched that non-uniformly of the tension, due small deflections can be neglected

If a unit concentrated load is applied y direction at an arbitrary point ξ, the string will be deflected into two linear parts with corner at the point $x=\xi$

If we denote the uniform tension required force equilibrium in the y direction to the condition $T\sin\theta_1 + T\sin\theta_2 = 1$ --- (1)

For small directions (and we have.

$$\left\{ \begin{array}{l} \sin\theta_1 \approx \tan\theta_1 = \dfrac{\delta}{\xi} \\ \sin\theta_2 \approx \tan\theta_2 = \dfrac{\delta}{l-\xi} \end{array} \right\} \text{--- (2)}$$

maximum deflection of the string at the loaded

Sub (2) in (1) we have

$$T\frac{\delta}{\xi} + T\frac{\delta}{x-\xi} = 1$$

Suppose that the string is rotated uniformly about the x-axis, with angular velocity ω and that in addition a continuous distribution of loading force is imposed in the direction radially outward from the axis of revolution

If the linear mass density of the string is denoted by $\rho(x)$, then the total effective and load intensity can be written in the form

$$P(x) = \omega^2 \rho(x) y(x) + f(x)$$

Sub $p(x)$ in (6) we get

$$y(x) = \int_0^l G(x,\xi) \left[\omega^2 \rho(\xi) y(\xi) + f(\xi)\right] d\xi$$

$$= \omega^2 \int_0^l G(x,\xi) \rho(x) y(\xi) d\xi + \int_0^l G(x,\xi) f(\xi) d\xi$$

Problem: 2

Prove that eqn $\omega^2 \int_0^l G(x,\xi) \rho(x) y(x) d\xi + \int_0^l G(x,\xi) f(x) d\xi$ to the differential equation with boundary condition $T \frac{d^2y}{dx^2} \rho \omega^2 y + f = 0$, $y(0) = 0$ & $y(l) = 0$

Soln

Given $y(x) = \omega^2 \int_0^l G(x,\xi) \rho(\xi) y(\xi) d\xi + \int_0^l G(x,\xi) f(\xi) d\xi$

where $G(x,\xi) = \begin{cases} \frac{x(l-\xi)}{Tl} & \text{if } x < \xi \\ \frac{\xi(l-x)}{Tl} & \text{if } x > \xi \end{cases}$

$$y(x) = \omega^2 \int_0^x G(x,\xi) \rho(\xi) y(\xi) d\xi + \omega^2 \int_x^l G(x,\xi) \rho(\xi) y(\xi) d\xi$$
$$+ \int_0^x G(x,\xi) f(\xi) d\xi + \int_x^l G(x,\xi) f(\xi) d\xi$$

$$y = \omega^2 \left[\int_0^x \frac{\xi(l-x)}{Tl} \rho(\xi) y(\xi) d\xi + \int_x^l \frac{x(l-\xi)}{Tl} \rho(\xi) d\xi \, y(\xi) \right.$$

$$T\delta\left(\frac{1}{\epsilon}+\frac{1}{l-\epsilon}\right)=1$$

$$T\delta\left(\frac{l-\epsilon+\epsilon}{\epsilon(l-\epsilon)}\right)=1$$

$$T\delta\left(\frac{l}{\epsilon(l-\epsilon)}\right)=1$$

$$\delta=\frac{\epsilon(l-\epsilon)}{Tl} \quad \text{--- (3)}$$

when $x<\epsilon$ the corresponding points are $(0,0)$ & (ϵ,δ)

then the eqn is $\frac{x-0}{\epsilon-0}=\frac{y-0}{\delta-0}$ $\left[\because \frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}\right]$

$$\frac{x}{\epsilon}=\frac{y}{\delta} \Rightarrow y=\frac{x\delta}{\epsilon} \text{ if } x<\epsilon$$

when $x>\epsilon$ the corresponding points are $(l,0)$ & (ϵ,δ)

Then the eqn is $\frac{x-l}{\epsilon-l}=\frac{y-0}{\delta-0}$

$$\Rightarrow \frac{y}{\delta}=\frac{x-l}{\epsilon-l}=\frac{-(l-x)}{-(l-\epsilon)}$$

$$y=\delta\left(\frac{l-x}{l-\epsilon}\right)$$

Hence the eqn of the deflection curve is

$$y=\begin{cases}\delta\frac{x}{\epsilon} & \text{if } x<\epsilon \\ \frac{\delta(l-x)}{(l-\epsilon)} & \text{if } x>\epsilon\end{cases} \quad \text{--- (4)}$$

Sub the values of δ in (4) we get

$$G(x,\delta)=\begin{cases}\frac{\epsilon(l-\epsilon)x}{\epsilon\cdot Tl} & \text{if } x<\epsilon \\ \frac{\epsilon(l-\epsilon)(l-x)}{(l-\epsilon)T\cdot l} & \text{if } x>\epsilon\end{cases}$$

Hence the influence function is given by

$$=\begin{cases}\frac{x(l-\epsilon)}{Tl} & \text{if } x<\xi \\ \frac{\xi(l-x)}{Tl} & \text{if } x>\xi\end{cases} \quad \text{--- (5)}$$

the deflection $y(x)$ due to a loading distribution $P(x)$ is given by , $y(x)=\int^{l} G(x,\epsilon)\, P(\epsilon)\, d\epsilon$ --- (6)

$$\frac{dy}{dx} = \omega^2\left[\frac{(l-x)}{Tl}\left(\xi\, p(\xi)\, y(\xi)\right]_0^x + \frac{\xi}{Tl}\int_0^x p(\xi)\, y(\xi)\,(-1)\,d\xi\right]$$

$$+\,\omega^2\left[\frac{x(l-\xi)}{Tl}\left[p(\xi)\, y(\xi)\right]_x^l + \int_x^l \frac{(l-\xi)}{Tl}\, p(\xi)\, y(\xi)\, d\xi\right]$$

$$+\left[\frac{(l-x)}{Tl}\left[\xi f(\xi)\right]_0^x + \int_0^x \frac{\xi f(\xi)}{Tl}(-1)\,d\xi\right] + \left[\frac{x(l-\xi)}{Tl}\left[f(\xi)\right]_x^l\right.$$

$$\left. + \int_x^l \frac{(l-\xi)}{Tl}\, f(\xi)\, d\xi\right]$$

$$= \omega^2\left[\int_0^x \frac{-\xi}{Tl}\, p(\xi)\, y(\xi)\, d\xi + x\, p(x)\, y(x)\,\frac{(l-x)}{Tl} + \int_x^l \frac{(l-\xi)}{Tl}\, p(\xi)\, y(\xi)\, d\xi\right.$$

$$\left. - \frac{x(l-x)}{Tl}\, p(x)\, y(x)\right] + \int_0^x \frac{-\xi}{Tl}\, f(\xi)\, d\xi + \frac{x(l-x)}{Tl}\, f(x)$$

$$+ \int \frac{(l-\xi)}{Tl}\, f(\xi)\, d\xi - \frac{x(l-x)}{Tl}\, f(x)$$

$$\frac{dy}{dx} = \omega^2\left[\int_0^x -\frac{\xi}{Tl}\, p(\xi)\, y(\xi)\, d\xi + \int_x^l \frac{(l-\xi)}{Tl}\, p(\xi)\, y(\xi)\, d\xi\right]$$

$$+ \int_0^x \frac{-\xi}{Tl}\, f(\xi)\, d\xi + \int_x^l \frac{(l-\xi)}{Tl}\, f(\xi)\, d\xi$$

Now

$$\frac{d^2y}{dx^2} = -\omega^2\left[\frac{\xi}{Tl}\, p(\xi)\, y(\xi)\right]_0^x + \omega^2\left[\frac{(l-\xi)}{Tl}\, p(\xi)\, y(\xi)\right]_x^l$$

$$-\left[\frac{\xi}{Tl}\, f(\xi)\right]_0^x + \left[\frac{(l-\xi)}{Tl}\, f(\xi)\right]_x^l$$

$$= -\frac{\omega^2}{Tl}\left[x\, p(x)\, y(x)\right] - \frac{\omega^2}{Tl}\left[(l-x)\, p(x)\, y(x)\right] - \frac{1}{Tl}\left[x\, f(x)\right]$$

$$\frac{1}{Tl}\left[(l-x)\, f(x)\right]$$

$$= \frac{+\omega^2}{Tl}\left[-x\, p(x)\, y(x) + (l-x)\, p(x)\, y(x)\right] + \frac{1}{Tl}\left[-x f(x)\right.$$

$$\left. -(l-x) f(x)\right]$$

$$= \frac{\omega^2}{Tl}\left[-x\, p(x)\, y(x) - l\,(p(x)\, y(x)) + x\, p(x)\, y(x)\right]$$

$$+ \frac{1}{Tl}\left[-x f(x) - l f(x) + x f(x)\right]$$

$$= \frac{\omega^2}{Tl}\left[-l\, p(x)\, y(x)\right] + \frac{1}{Tl}\left[-l\, f(x)\right]$$

$$\therefore \frac{d^2y}{dx^2} = \frac{-\omega^2}{T} f(x)\, y(x) = \frac{-f(x)}{T}$$

multiply by T, $T \dfrac{d^2y}{dx^2} = -\omega^2 f(x) y(x) - f(x)$

$$T \frac{d^2y}{dx^2} + \omega^2 f(x)\, y(x) + f(x) = 0$$

$$y(0) = 0,\ y(l) = 0$$

Sec 8.6.

Fredohlm eqns with Separable Kernals

Defn:

A kernal $K(x, \xi)$ separable if it can be exp as the sum of finite no of terms each of the te is the product of a function of x alone and a of ξ alone such of a kernal is expressed in th following form

$$K(x, \xi) = \sum_{n=1}^{N} f_n(x)\, g(\xi)$$

Note:

In the above defn, there is without loss of generality if we assume that N function f are linearly independent Any polynomial in x a ξ is of this type

Example:

the kernal $\sin(x-\xi)$ is separable

since $\sin(x+\xi) = \sin x \cos \xi + \sin \xi \cos x$

Remark:

Integral eqns with separable kernal do occur frequently in practice However, th are easily treated and furthermore, the result their consideration lead to a integral eqn of general type

Also it is possible to apply the method to be develop on this section to the appropriate sdn of fredholm eqn is which the kernal can be satisfactionary approximal by a polynomial in x & or a separable of more general.

Defn:

The integral equation is said to be homogenous the function $f(x)=0$

(i.e) $y(x)=\lambda\int_a^b k(x,t)\,y(t)\,dt$

Defn:

The value of λ for which $\Delta(\lambda)=0$ are know as the characteristic value (or) Eigen values

Defn:

Any non-trivial solution of the homogenous corresponding characteristic integral eqn is called a function of the integral function (or) eigen valued

eqn

Note:

If k of the constant $c_1, c_2, \ldots c_N$ can be assigned arbitrarily for a given characteristic value of λ then k linearly independent function are obtained

Defn:

The integral eqn is said to be incompatible if there is no solution exist

Defn: The integral eqn is said to be redundant if there are finite many solution exists.

Problem:

Determine the solution of the fredholm equation with separable kernals.

Solution:

Consider the fredholm equation of the second kind with separable kernal.

Then it can be written as,

$$y(x) = F(x) + \lambda \int_a^b k(x, \xi)\, y(\xi)\, d\xi \longrightarrow (1)$$

$$= F(x) + \lambda \int_a^b \left[\sum_{n=1}^{N} f_n(x)\, g_n(\xi) \right] y(\xi)\, d\xi \quad [\because k(x,\xi) \text{ is separable}]$$

$$= F(x) + \lambda \sum_{n=1}^{N} f_n(x) \int_a^b g_n(\xi)\, y(\xi)\, d\xi \longrightarrow (2)$$

The co-efficient of $f_1(x), f_2(x), \ldots, f_N(x)$ in the above equations are constants.

Although their values are unknown.

let $C_n = \int_a^b g_n(x)\, y(x)\, dx \quad (n = 1 \text{ to } N)$

$$(2) \Rightarrow y(x) = F(x) + \lambda \sum_{n=1}^{N} f_n(x)\, c_n \longrightarrow (3)$$

This is the form of the required solution of the integral equation (1) and it remains only to determine the N constants $c_1, c_2, \ldots, c_N$.

Now, on multiplying the equation (3) by $g_1(x)$ and integrating over the integral (a, b) we get,

$$\int_a^b g_1(x)\, y(x)\, dx = \int_a^b g_1(x) F(x)\, dx + \lambda \sum_{n=1}^{N} c_n \int_a^b f_n(x)\, g_1(x)\, dx$$

$$\Rightarrow c_1 = \int_a^b g_1(x)F(x)\,dx + \lambda \sum_{n=1}^{N} c_n \int_a^b f_n(x) g_1(x)\,dx.$$

Now, on multiplying the equation (3) by $g_2(x)$ & integrating over the integral (a,b) we have.

$$\int_a^b g_2(x)y(x)\,dx = \int_a^b g_2(x)F(x)\,dx + \lambda \sum_{n=1}^{N} c_n \int_a^b f_n(x) g_2(x)\,dx$$

$$\Rightarrow c_2 = \int_a^b g_2(x)F(x)\,dx + \lambda \sum_{n=1}^{N} c_n \int_a^b f_n(x) g_2(x)\,dx.$$

proceeding in this way, we have.

$$c_N = \int_a^b g_N(x)F(x)\,dx + \lambda \sum_{n=1}^{N} c_n \int_a^b f_n(x) g_N(x)\,dx$$

let $\alpha_{mn} = \int_a^b f_n(x) g_m(x)\,dx$ and $\beta_m = \int_a^b g_m(x) F(x)\,dx.$

$\therefore c_1 = \beta_1 + \lambda \sum_{n=1}^{N} c_n \alpha_{1n}, \quad c_2 = \beta_2 + \lambda \sum_{n=1}^{N} c_n \cdot \alpha_{2n},$

and so on. $c_N = \beta_N + \lambda \sum_{n=1}^{N} c_n \alpha_{Nn}$

$$\Rightarrow c_1 = \beta_1 + \lambda c_1 \alpha_{11} + \lambda c_2 \alpha_{12} + \cdots + \lambda c_N \alpha_{1N}$$

$$c_2 = \beta_2 + \lambda c_1 \alpha_{21} + \lambda c_2 \alpha_{22} + \cdots + \lambda c_N \alpha_{2N}$$

$$\vdots$$

$$c_N = \beta_N + \lambda c_1 \alpha_{N1} + \lambda c_2 \alpha_{N2} + \cdots + \lambda c_N \alpha_{NN}$$

$$\left.\begin{aligned} \Rightarrow (1-\lambda\alpha_{11})c_1 - \lambda \alpha_{12} c_2 - \cdots - \lambda \alpha_{1N} c_N &= \beta_1 \\ -\lambda \alpha_{21} c_1 + (1-\lambda\alpha_{22}) c_2 - \cdots - \lambda \alpha_{2N} c_N &= \beta_2 \\ \vdots \\ -\lambda \alpha_{N1} c_1 - \lambda \alpha_{N2} c_2 - \cdots - (1-\lambda\alpha_{NN}) c_N &= \beta_N \end{aligned}\right\} \longrightarrow (4).$$

$$\begin{pmatrix} (1-\lambda\alpha_{11}) & -\lambda\alpha_{12} & \cdots & -\lambda\alpha_{1N} \\ -\lambda\alpha_{21} & (1-\lambda\alpha_{22}) & \cdots & -\lambda\alpha_{2N} \\ \vdots & & & \\ -\lambda\alpha_{N1} & -\lambda\alpha_{N2} & \cdots & (1-\lambda\alpha_{NN}) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_N \end{pmatrix}$$

$\Rightarrow (I - \lambda A)c = \beta$ where I is the unit matrix of order N

$$A = [a_{ij}] = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & & & \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{pmatrix}, \quad c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{pmatrix} \ \& \ \beta = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_N \end{pmatrix}$$

Case : (i)

Suppose $F(x) = 0$ in (1).

Then the integral equation is said to be homogeneous and obviously satisfied by the trivial solution $y(x) = 0$, corresponds to the trivial solution $c_1 = c_2 = \cdots = c_N = 0$, when $\beta = 0$.

This is the only solution of determinant $\Delta = |I - \lambda A| \neq 0$. If $\Delta = 0$, atleast one of the c's can be assigned arbitrary and the remaining c's be determined accordingly.

Thus we have infintely many solution of the integral equation (1).

Case : (ii)

Suppose $F(x) \neq 0$

But its orthogonal to all function $g_1(x), g_2(x) \ldots g_N(x)$

Then $\beta = 0$.

$\therefore$ By proceeding discussion again applies to the case, except for the fact that the solution (3) of the integral equation involves also the function $F(x)$.

The trivial values $c_1 = c_2 = \cdots c_N = 0$ is the solution of $y = F(x)$.

Solution corresponding to characteristic values of λ are expressed as the sum of $F(x)$ and arbitrary multiples of characteristic functions.

Suppose atleast one right hand member of (4) does not vanish a unique non-trivial Solution of the integral equation (1) is the determinant $\Delta(\lambda) \neq 0$.

If $\Delta\lambda = 0$ equation (4) are either incompatiable and no Solution exists or they are redudent and infinitely many Solution exist.

3.7. Illustractive Examples.

Problem:

obtain the general Solution of the integral equation

$$y(x) = \lambda \int_0^1 (1 - 3x\xi)\, y(\xi)\, d\xi + F(x).$$

Solution:

$$\text{Given } y(x) = \lambda \int_0^1 (1 - 3x\xi)\, y(\xi)\, d\xi + F(x) \longrightarrow (1).$$

$$= \lambda \int_0^1 y(\xi)\, d\xi - \lambda \int_0^1 3x\xi\, y(\xi)\, d\xi + F(x)$$

$$y(x) = \lambda c_1 - \lambda 3x c_2 + F(x) \longrightarrow (2)$$

where $c_1 = \int_0^1 y(\xi)\, d\xi$ and $c_2 = \int_0^1 \xi\, y(\xi)\, d\xi$

Multiply (2) by 1 and integrate the result w.r.to 'x' over $(0,1)$, we get,

$$\int_0^1 y(x)\, dx = \lambda \int_0^1 (c_1 - 3x c_2)\, dx + \int_0^1 F(x)\, dx.$$

$$c_1 = \lambda \left[c_1 x - \frac{3x^2}{2} c_2 \right]_0^1 + \int_0^1 F(x)\, dx$$

$$c_1 = \lambda \left(c_1 - \frac{3c_2}{2} \right) + \int_0^1 F(x)\, dx \longrightarrow (3).$$

Multiply ② by x and integrate the result w.r.to x over $(0,1)$, we get $\int_0^1 x\, y(x)\,dx = \lambda \int_0^1 [c_1 x - 3x^2 c_2]\,dx + \int_0^1 x\, F(x)\,dx$

$$c_2 = \lambda \left[c_1 \frac{x^2}{2} - \frac{3x^3}{3} c_2 \right]_0^1 + \int_0^1 x\, F(x)\,dx$$

$$c_2 = \lambda \left(\frac{1}{2} c_1 - c_2 \right) + \int_0^1 x\, F(x)\,dx \longrightarrow ④$$

$$③ \Rightarrow (1-\lambda)c_1 + \frac{3}{2}\lambda c_2 = \int_0^1 F(x)\,dx \longrightarrow ⑤$$

$$④ \Rightarrow \left(\frac{-1}{2}\right)\lambda c_1 + (1-\lambda) c_2 = \int_0^1 x\, F(x)\,dx \longrightarrow ⑥$$

Now, the determinants of co-efficient is

$$\Delta(\lambda) = \begin{vmatrix} (1-\lambda) & 3/2\,\lambda \\ -1/2\,\lambda & 1+\lambda \end{vmatrix}$$

$$= (1-\lambda)(1+\lambda) - \left(3/2\,\lambda\right)\left(-1/2\right)\lambda$$

$$= 1 + \lambda - \lambda - \lambda^2 + \frac{3}{4}\lambda^2 = 1 - \lambda^2 + 3/4\,\lambda^2$$

$$= \frac{1}{4}(4 - \lambda^2)$$

$\therefore \exists$ a unique solution $\Delta(\lambda) = \frac{1}{4}(4-\lambda^2)$

$\Leftrightarrow \lambda \neq \pm 2$.

If $\Delta(\lambda) = 0$, then $\lambda = \pm 2$.

When $\lambda = 2$.

$$⑤ \Rightarrow -c_1 + 3c_2 = \int_0^1 F(x)\,dx \longrightarrow ⑦$$

$$⑥ \Rightarrow -c_1 + 3c_2 = \int_0^1 x\, F(x)\,dx \longrightarrow ⑧$$

When $\lambda = -2$.

$$⑤ \Rightarrow 3c_1 - 3c_2 = \int_0^1 F(x)\,dx$$

$$\Rightarrow c_1 - c_2 = \frac{1}{3}\int_0^1 F(x)\,dx \longrightarrow ⑨$$

$$⑥ \Rightarrow c_1 - c_2 = \int_0^1 x\, F(x)\,dx \longrightarrow ⑩$$

$\therefore$ (7) and (8) are incompatible unless the prescribed function $F(x)$ satisfied the condition.

$$\int_0^1 F(x)\,dx = \int_0^1 x\,F(x)\,dx$$

$$\Rightarrow \int_0^1 (1-x)F(x)\,dx = 0 \longrightarrow (11).$$

(9) and (10) are incompatible unless $F(x)$ satisfied the condition $\frac{1}{3}\int_0^1 F(x)\,dx = \int_0^1 x\,F(x)\,dx$.

$$\Rightarrow \int_0^1 \left(\frac{1}{3} - x\right) f(x)\,dx = 0.$$

$$\Rightarrow \int_0^1 (1-3x) f(x)\,dx = 0 \longrightarrow (18).$$

When $\lambda = \pm 2$, there exist infinitely many solution for the given integral equation.

Case :(i)

Suppose $F(x) = 0$.

Then (1) is homogeneous equation.

If $\lambda \neq \pm 2$, then the only solution is a trivial one $y(x) = 0$.

If $\lambda = 2$ (and $F = 0$), then $c_1 = 3c_2$ [$\because$ by (8) & (7)]

Then (8) $\Rightarrow y(x) = 2(3c_2 - 3x c_2)$

$= 6c_2(1-x)$

$= A(1-x)$ [$\because$ characteristic function corresponding to $\lambda = 2$]

If $\lambda = -2$, then $c_1 = c_2$ (by (9) & (10))

(8) $\Rightarrow y(x) = -2(c_1 - 3x c_1)$

$= -2c_1(1-3x) = B(1-3x)$.

This is a characteristic function corresponding to $\lambda = 2$.

$\therefore$ The general solution is, $y(x) = F(x) + A_1(1-x) + B_2(1-3x)$

$\Rightarrow \lambda(c_1 - 3x c_2) + F(x) = F(x) + A_1(1-x) + B_2(1-3x)$ ($\because$ by 8)

$\Rightarrow -3c_2\lambda = -A_1 - 3B_2$ [$\because$ equating the co-efficients of x]

Equating the constant term, $\lambda c_1 = A_1 + B_2 \longrightarrow$ (14)

$$-\cancel{A_1} - 3B_2 = -3c_2\lambda$$
$$A_1 + B_2 = \lambda c_1$$
$$-2B_2 = \lambda(c_1 - 3c_2)$$

$\Rightarrow B_2 = \frac{\lambda}{2}(3c_2 - c_1)$

Sub in (14), $A_1 + B_2 = \lambda c_1$

$\Rightarrow A_1 + \frac{\lambda}{2}(3c_2 - c_1) = \lambda c_1$

$\Rightarrow A_1 + \frac{3}{2}\lambda c_2 = \lambda c_1 + \frac{\lambda c_1}{2}$

$\Rightarrow A_1 = \lambda c_1 + \frac{\lambda c_1}{2} - \frac{3\lambda c_2}{2}$

$= -\frac{3\lambda}{2}c_2 + \frac{3}{2}\lambda c_1$

$A_1 = \frac{3\lambda}{2}(c_1 - c_2)$

The general solution is $y(x) = F(x) + A_1(1-x) + B_2(1-3x)$

where $A_1 = \frac{3}{2}\lambda(c_1 - c_2)$, $B_2 = \frac{\lambda}{2}(3c_2 - c_1)$

Case : (ii)

Suppose $F(x) \neq 0$

Then there exist a unique solution $\lambda \neq \pm 2$.

If $\lambda = 2$, then (11) shows that no solution exist unless $F(x)$ is orthogonal to $(1-x)$ over $(0,1)$ in which case infinitely many solution exist.

From (7) & (8), $c_1 = 3c_2 - \int_0^1 F(x)\,dx$.

$\therefore$ (2) $\Rightarrow y(x) = F(x) + 2\left[3c_2 - \int_0^1 F(x)\,dx - 3x c_2\right]$

$= F(x) + 6c_2(1-x) - 2\int_0^1 f(x)\,dx$

$= F(x) + A(1-x) - 2\int_0^1 F(x)\,dx$, where $A = 6c_2$ is a

Constant when $\int_0^1 (1-x)F(x)\,dx = 0$.

If $\lambda = -2$, then (12) shows that no solution exist unless $F(x)$ is orthogonal to $(1-3x)$ over $(0,1)$ in which case infinitely many solution exist.

From, (9) & (10), $c_1 = c_2 + \frac{1}{3}\int_0^1 F(x)\,dx$

Sub c_1 in (8), $y(x) = F(x) - 2\left[c_2 + \frac{1}{3}\int_0^1 F(x)\,dx - 3x c_2\right]$

$= F(x) + B(1-3x) - \frac{2}{3}\int_0^1 F(x)\,dx.$

where $B = -2c_2$ is a constant term when

$\int_0^1 (1-3x)F(x)\,dx = 0$.